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Periodic Solutions for Second-Order Differential Equations involving Nonconvex Superpotentials

SAMIR ADLY[1] and DUMITRU MOTREANU[2]
[1]*LACO, Université de Limoges, Limoges, France*
[2]*Universitatea Al. I. Cuza, Department of Mathematics, Iasi, Romania*
E-mail: motreanu@uaic.ro

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Abstract. The paper establishes the existence of a nonconstant periodic solution of a general second order nonautonomous Hamiltonian system with discontinuous nonlinearities. The multiplicity of solutions is also studied.

Key words: Discontinuous Hamiltonian system; Generalized gradient; Hemivariational inequality; Periodic solution

1. Introduction and the statement of the main result

In recent years, the theory of variational inequalities has been considerably developed. It is well known that these unilateral problems express the principle of virtual works or powers in its inequality form and are closely connected with the convexity of the corresponding energy functionals involved, i.e., on the monotonicity of their gradients (in the smooth case) or their subdifferentials (in the nonsmooth one).

In the case of the lack of convexity, the variational expression of the problem leads to a new type of formulation called hemivariational inequality theory introduced and developed by P.D. Panagiotopoulos since the early 1980s. Hemivariational inequalities are derived from nonconvex and nondifferentiable superpotentials by using the mathematical notion of generalized gradient of F.H. Clarke [3] for locally Lipschitz functions. The hemivariational inequality approach has now been proved to be very efficient to describe the behaviour of several mechanical problems, e.g., the delamination problem of multilayered plates, nonmonotone semipermeability problems, the partial debonding of adhesive joints, etc. For more details concerning this approach, we refer the readers to the book of Panagiotopoulos [7].

The paper is devoted to the following boundary value problem, denoted ($\mathscr{P}$), for a second order nonautonomous Hamiltonian system with discontinuous nonlinearities in $\mathbb{R}^N$:

$$-\ddot{u}(t)+\xi(t)=e(t)$$
$$\xi(t)\in\partial_x j(t,u(t)) \quad \text{a.e. } t\in(0,T)\,,$$
$$u(T)-u(0)=\dot{u}(T)-\dot{u}(0)=0\,,$$

for a prescribed period $T>0$.

By a solution of problem $(\mathscr{P})$ we mean a function $u\in W^{2,1}(0,T;\mathbb{R}^N)$ for which one finds a function $\xi\in L^1(0,T;\mathbb{R}^N)$ provided the relations in $(\mathscr{P})$ are verified. Let us precise now the meaning of the data entering problem $(\mathscr{P})$. Throughout the paper, the norm of the Euclidean space $\mathbb{R}^N$ is denoted by $|\cdot|$. Let $e\in L^2(0,T;\mathbb{R}^N)$. The function $j:\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is required to satisfy the following conditions:

(j_0) for each $x\in\mathbb{R}^N$, $j(\cdot\,,x)$ is a measurable, T-periodic function, and $j(t,0)=0$, $\forall t\in\mathbb{R}$;

(j_1) for every constant $M>0$ there is $k_M\in L^1(0,T;\mathbb{R})$ such that

$$|j(t,x)-j(t,y)|\leqslant k_M(t)|x-y|\,,\quad \forall|x|,|y|\leqslant M\,,\quad \text{a.e. } t\in(0,T)\,;$$

(j_2) there exist numbers $\mu>2$, $\alpha>0$, $\beta>0$ such that

$$j(t,x)-\frac{1}{\mu}j^0(t,x;x)\geqslant\alpha|x|^2-\beta\,,\quad \forall x\in\mathbb{R}^N\,,\quad \text{a.e. } t\in(0,T)\,,$$

where $j^0(t,x;y)$ stands for the generalized directional derivative of $j(t,\cdot)$ at x in the direction y and is defined by the formula

$$j^0(t,x;y):=\limsup_{\substack{\lambda\to 0^+\\ z\to x}}\frac{j(t,z+\lambda y)-j(t,z)}{\lambda}\,;$$

(j_3) there exists a point $x_0\in\mathbb{R}^N$, with $|x_0|\geqslant(\beta/\alpha)^{1/2}$, such that

$$\int_0^T j(t,x_0)\,dt<0\,;$$

(j_4) there exist numbers $r>0$ and $\gamma>0$ such that

$$j(t,x)\geqslant e(t)\cdot x+\gamma|x|^2\,,\quad \forall|x|\leqslant r\,,\quad \text{a.e. } t\in(0,T)\,;$$

(j_5) if $y\in\mathbb{R}^N$ satisfies $e(t)\in\partial_x j(t,y)$ for a.e. $t\in(0,T)$, then

$$\int_0^T j(t,y)\,dt\leqslant\int_0^T e(t)\,dt\cdot y\,;$$

(j_6) for all $t\in\mathbb{R}$, there is a continuous function $\alpha(t)>0$ such that

$$j(t,x)\leqslant-\alpha(t)|x|^\nu \text{ for each } |x|\geqslant\bar{r}\,,\quad x\in\mathbb{R}^n\,,$$

where $\nu>2$ and $\bar{r}>0$.

Here the notation $\partial_x j(t,\cdot)$ stands for the generalized gradient of $j(t,\cdot)$ in the sense of Clark which is defined by:

$$\partial_x j(t, x) := \{x^* \in \mathbb{R}^N \mid j^0(t, x; y) \geqslant \langle x^*, y \rangle , \quad \forall y \in \mathbb{R}^N\} .$$

The main difficulty in the study of problem ($\mathscr{P}$) consists in the fact that the corresponding potential is nonconvex, nonsmooth, without the usual coercive quadratic term in the variable $u(t)$ and under the presence of the linear term determined by the function $e(t)$ (compare, e.g., with the nonsmooth and nonconvex Hamiltonian system treated in Adly, Goeleven and Motreanu [1]). This does not allow to apply directly to problem ($\mathscr{P}$) the approach relying on mini–max results in the nonsmooth critical point theory (see Chang [2] and Motreanu [4]) because it does not fit into the superlinear setting of such framework. We overcome here these difficulties by imposing suitable growth conditions for the nonlinear part $j(t, x)$ in order to compensate the lack of coerciveness of the linear part (see assumptions (j_0)–(j_5)).

The aim of our study is to provide verifiable sufficient conditions ensuring the existence of nontrivial solutions to problem ($\mathscr{P}$). Our existence result concerning problem ($\mathscr{P}$) is the following.

THEOREM 1.1. *Under assumptions (j_0)–(j_5), problem ($\mathscr{P}$) has a nonconstant solution $u \in W^{2,1}(0, T; \mathbb{R}^N)$ with u and $\dot{u}$ periodic.*

The theorem above is closely related to the theory of hemivariational inequalities. For the formulation of problem ($\mathscr{P}$) in terms of hemivariational inequalities as well as for related results, comments and applications we refer to [1, 5–7]. Finally, we establish in Theorem 3.1 a multiplicity result for the set of solutions to problem ($\mathscr{P}$). In fact, it is shown that under the additional hypothesis (j_6), assuming that the potential $j(t, \cdot)$ is even on $\mathbb{R}^N$, for each t, and $e = 0$, problem ($\mathscr{P}$) admits infinitely many nonconstant solutions.

The rest of the paper is organized as follows. Section 2 contains the proof of Theorem 1.1. Section 3 deals with the multiplicity result for problem ($\mathscr{P}$).

2. Proof of the main result

In this section we give the proof of Theorem 1.1. For a later use we introduce the functional $J : L^\infty(0, T; \mathbb{R}^N) \to \mathbb{R}$ by

$$J(u) = \int_0^T j(u(t))\, dt , \quad \forall u \in L^\infty(0, T; \mathbb{R}^N) .$$

Let us show that the functional J is Lipschitz continuous on bounded subsets of $L^\infty(0, T; \mathbb{R}^N)$. To this end let $M > 0$ be a fixed number. Then, by assumption (j_1), we can write

$$\begin{aligned}|J(v) - J(w)| &\leqslant \int_0^T |j(t, v(t) - j(t, w(t))| \, dt \\ &\leqslant \int_0^T k_M(t)|v(t) - w(t)| \, dt \\ &\leqslant \|k_M\|_{L^1}\|v - w\|_{L^\infty} .\end{aligned}$$

This justifies that J is Lipschitz continuous on bounded sets in $L^\infty(0, T; \mathbb{R}^N)$. Consequently, it makes sense to consider the generalized gradient $\partial J(u)$ of J at any $u \in L^\infty(0, T; \mathbb{R}^N)$.

Moreover, one can check readily that assumption (j_1) implies the next condition in Clarke [2], p. 80: for every $u \in L^\infty(0, T; \mathbb{R}^N)$ there exist $\varepsilon > 0$ and $k \in L^1(0, T; \mathbb{R})$ such that $|j(t, x) - j(t, y)| \leqslant k(t)|x - y|$ whenever $|x - u(t)|$, $|y - u(t)| \leqslant \varepsilon$, a.e. $t \in (0, T)$.

This allows to conclude that for every $z \in \partial J(u)$ there exists $\xi = \xi(z) \in L^1(0, T; \mathbb{R}^N)$ (in fact, $\xi(\cdot) \cdot v(\cdot) \in L^1(0, T; \mathbb{R}^N)$, $\forall v \in L^\infty(0, T; \mathbb{R}^N)$) satisfying

$$\langle z, v \rangle = \int_0^T \xi(t) \cdot (t) \, dt \,, \quad \forall v \in L^\infty(0, T; \mathbb{R}^N) \tag{2.1}$$

and

$$\xi(t) \in \partial_x j(t, u(t)) \quad \text{a.e. } t \in (0, T) \tag{2.2}$$

(see Clarke [2], p. 80).

Let H_T^1 denote the Hilbert space of T-periodic, absolutely continuous, $\mathbb{R}^N$-valued functions on $\mathbb{R}$ whose derivative is square integrable on $(0, T)$. The Hilbert space H_T^1 is endowed with the norm

$$\|u\|^2 = \int_0^T (|u(t)|^2 + |\dot{u}(t)|^2) \, dt) \,, \quad \forall u \in H_T^1 \,.$$

By means of the functional J we introduce $I : H_T^1 \to \mathbb{R}$ by

$$\begin{aligned}I(u) &= \frac{1}{2} \int_0^T |\dot{u}(t)|^2 \, dt + \int_0^T j(t, u(t)) \, dt - \int_0^T e(t) \cdot u(t) \, dt \\ &= \frac{1}{2} \|\dot{u}\|_{L^2}^2 + J(u) - (e, u)_{L^2} \,, \quad \forall u \in H_T^1 \,.\end{aligned} \tag{2.3}$$

It is clear that the functional I is locally Lipschitz on H_T^1. We apply to I the Mountain Pass Theorem in Chang's variant for locally Lipschitz functions (see Chang [2]).

Firstly, we check that $I : H_T^1 \to \mathbb{R}$ satisfies the Palais-Smale condition in the sense of Chang [2] for locally Lipschitz functionals. Towards this let $\{u_n\} \subset H_T^1$ be a sequence such that $I(u_n)$ is bounded, say

$$|I(u_n)| \leqslant c_0 \,, \quad n \geqslant 1 \,, \tag{2.4}$$

and there is $w_n \in \partial I(u_n)$ with

$$w_n \to 0 \text{ in } H_T^{1*} \text{ as } n \to \infty . \tag{2.5}$$

We claim that

$$\{u_n\} \text{ is bounded in } H_T^1 . \tag{2.6}$$

Taking into account (2.4) and (2.5), for n sufficiently large, we can write

$$\begin{aligned} c_0 + \frac{1}{\mu}\|u_n\| &\geqslant c_0 + \frac{1}{\mu}\|u_n\|\,\|w_n\|_* \\ &\geqslant I(u_n) - \frac{1}{\mu}\langle w_n, u_n\rangle \\ &\geqslant \left(\frac{1}{2}-\frac{1}{\mu}\right)\|\dot u_n\|_{L^2}^2 + \int_0^T \left(j(t,u_n) - \frac{1}{\mu} z_n \cdot u_n\right) dt \\ &\quad + \left(\frac{1}{\mu}-1\right)\int_0^T e\cdot u_n\, dt \\ &\geqslant \left(\frac{1}{2}-\frac{1}{\mu}\right)\|\dot u_n\|_{L^2}^2 + \int_0^T \left(j(t,u_n) - \frac{1}{\mu} j^0(t,u_n;u_n)\right) dt \\ &\quad + \left(\frac{1}{\mu}-1\right)\|e\|_{L^2}\|u_n\|_{L^2} , \end{aligned}$$

where $z_n(t) \in \partial_x j(t, u_n(t))$ for a.e. $t \in (0, T)$. Then, using assumption (j_2), one finds a constant c_1 such that

$$c_0 + \frac{1}{\mu}\|u_n\| \geqslant \left(\frac{1}{2}-\frac{1}{\mu}\right)\|\dot u_n\|_{L^2}^2 + \alpha\|u_n\|_{L^2}^2 - c_1 + \left(\frac{1}{\mu}-1\right)\|e\|_{L^2}\|u_n\|_{L^2} . \tag{2.7}$$

Since $\mu > 2$ and $\alpha > 0$, estimate (2.7) shows that property (2.6) is true.

The duality map $\Lambda : H_T^1 \to H_T^{1*}$ is given by

$$\Lambda = A + B , \tag{2.8}$$

with $A, B \in L(H_T^1, H_T^{1*})$ expressed as follows

$$\langle Au, v\rangle = \int_0^T \dot u \cdot \dot v\, dt$$

and

$$\langle Bu, v\rangle = \int_0^T u \cdot v\, dt ,$$

for all $u, v \in H_T^1$.

Denote by $i : H_T^1 \to L^\infty(0, T; \mathbb{R}^N)$ the inclusion map, which is known to be compact. The element $w_n \in \partial I(u_n)$ can be expressed as follows

$$w_n = Au_n + i^* z_n i - Be, \text{ for some } z_n \in \partial J(u_n) .$$

In view of (2.5) we know that

$$Au_n + i^*z_n i - Be \to 0 \text{ in } H_T^{1*} \text{ as } n \to \infty . \tag{2.9}$$

Property (2.6) yields that along a subsequence one has

$$u_n \to u \text{ in } L^2(0, T; \mathbb{R}^N) \text{ as } n \to \infty \tag{2.10}$$

for some $u \in H_T^1$. Again by (2.6) we derive that $\{u_n\}$ is a bounded sequence in $L^\infty(0, T; \mathbb{R}^N)$. Recall that the functional J is Lipschitz continuous on bounded subsets of $L^\infty(0, T; \mathbb{R}^N)$. This fact ensures the boundedness of the sequence $\{z_n\}$ in $L^\infty(0, T; \mathbb{R}^N)$. Then the compactness of the adjoint operator i^* enables us to obtain that, for a renamed subsequence, $\{i^*z_n i\}$ converges in H_T^{1*}. From (2.9) we see that $\{Au_n\}$ converges in H_T^{1*}. Combining with (2.10) we infer from (2.8) that $\{\Lambda u_n\}$ converges in H_T^{1*}. Consequently, we found a subsequence of $\{u_n\}$ denoted again by $\{u_n\}$ which is convergent in H_T^1. This completes the proof of Palais-Smale condition for the functional I.

The next step of the proof consists in showing the following estimate: there exists $\delta > 0$ such that

$$I(u) \geqslant \min\left\{\frac{1}{2}, \gamma\right\}\delta^2 , \quad \forall \|u\| = \delta . \tag{2.11}$$

In writing (2.11) we used the constant $\gamma > 0$ given in assumption (j_4).

Indeed, the continuity of the inclusion map $i : H_T^1 \to L^\infty(0, T; \mathbb{R}^N)$ provides a constant $c > 0$ such that

$$\|u\|_{L^\infty} \leqslant c\|u\| , \quad \forall u \in H_T^1 . \tag{2.12}$$

Then, setting $\delta = r/c$, with $r > 0$ given in (j_4), inequality (2.12) and assumption (j_4) imply that

$$I(u) \geqslant \min\left\{\frac{1}{2}, \gamma\right\}\|u\|^2 , \quad \forall \|u\| \leqslant \delta .$$

In particular, we get assertion (2.11).

In order to complete the justification of requirements in the nonsmooth version of Mountain Pass Theorem we need the following formula:

$$\partial_s(s^{-\mu}j(t, sx)) = -\mu s^{-\mu-1}j(t, sx) + s^{-\mu}\partial_x j(t, sx)x , \quad \forall t \in \mathbb{R}, \; x \in \mathbb{R}^N, \; s > 0 , \tag{2.13}$$

where the notation ∂_s stands for the generalized gradient with respect to the variable s. Lebourg's mean valued theorem and equality (2.13) ensure that for every $s > 1$ there is a $\tau \in (1, s)$ such that for every $t \in \mathbb{R}$, $x \in \mathbb{R}^N$ one has

$$s^{-\mu}j(t, sx) - j(t, x) \in (-\mu\tau^{-\mu-1}j(t, \tau x) + \tau^{-\mu}\partial_x j(t, \tau x)x)(s-1) .$$

It follows that

$$s^{-\mu}j(t, sx) - j(t, x) \leqslant \tau^{-\mu-1}(s-1)(j_x^0(t, \tau x; \tau x) - \mu j(t, \tau x)) ,$$

for all $t \in \mathbb{R}$, $x \in \mathbb{R}^N$, $s > 1$. On the basis of condition (j_2) one deduces that

$$j(t, sx) \leqslant s^{\mu} j(t, x) + \tau^{-\mu-1}(s-1)s^{\mu}\mu(-\alpha|\tau x|^2 + \beta)$$

for all $t \in \mathbb{R}$, $x \in \mathbb{R}^N$, $s > 1$. It turns out that

$$j(t, sx) \leqslant s^{\mu} j(t, x)\,, \quad \forall t \in \mathbb{R},\, s > 1,\, |x| \geqslant \left(\frac{\beta}{\alpha}\right)^{1/2}. \tag{2.14}$$

We point out that we can use in estimate (2.14) the point $x = x_0$ prescribed by hypothesis (j_3). Consequently, from (2.14) we obtain that

$$I(sx_0) = J(sx_0) - s\int_0^T e(t) \cdot x_0\, dt \leqslant s^{\mu}\int_0^T j(t, x_0)\, dt + s\|e\|_{L^1}|x_0|$$

whenever $s > 1$. Since $\mu > 2$, assumption (j_3) yields

$$I(sx_0) \to -\infty \text{ as } s \to +\infty\,. \tag{2.15}$$

Property (2.15) allows to conclude that for s sufficiently large one has that $I(sx_0) \leqslant 0$ and $\|sx_0\| > \delta$. Therefore all the assumptions of Mountain Pass Theorem for locally Lipschitz functionals are fulfilled in the case of function I on the Hilbert space H^1_T. Then there exists a point $u \in H^1_T$ such that

$$0 \in \partial I(u) \tag{2.16}$$

and

$$I(u) \geqslant \min\left\{\frac{1}{2}, \gamma\right\}\delta^2 > 0\,. \tag{2.17}$$

Comparing $I(0) = 0$ with (2.17), we derive that $u \neq 0$. Relation (2.16) reads as

$$\int_0^T \dot{u}(t) \cdot \dot{v}(t)\, dt + \int_0^T \xi(t) \cdot v(t)\, dt - \int_0^T e(t) \cdot v(t)\, dt = 0\,, \quad \forall v \in H^1_T\,. \tag{2.18}$$

From (2.18) we derive that $\dot{u}$ admits a weak derivative which is equal to

$$\ddot{u}(t) = \xi(t) - e(t) \text{ for a.e. } t \in (0, T)\,. \tag{2.19}$$

Hence, according to (2.19), (2.18) (taking v to be the vectors of the canonical basis in $\mathbb{R}^N$) and the regularity information $\xi \in L^1(0, T; \mathbb{R}^N)$, we derive that $u \in W^{2,1}(0, T; \mathbb{R}^N)$. By means of (2.18) and (2.19), it is seen that $\dot{u}$ satisfies

$$\dot{u}(T) = \dot{u}(0) + \int_0^T \ddot{u}(t)\, dt = \dot{u}(0) + \int_0^T (\xi(t) - e(t))\, dt = \dot{u}(0)\,.$$

We thus established that u fulfills the conditions of solving problem $(\mathscr{P})$.

It remains to check that the constructed solution u is not constant. Arguing by contradiction, let us admit that $u(t) = y$, $\forall t$. Then, by $(\mathscr{P})$, one has $e(t) \in \partial_x j(t, y)$ for a.e. $t \in (0, T)$, so one can apply hypothesis (j_5). Accordingly, by (j_5), we derive that

$$I(u) = I(y) = \int_0^T [j(t, y) - e(t) \cdot y]\, dt \leqslant 0 .$$

This contradicts relation (2.17). The proof of Theorem 1.1 is thereby completed.

3. A multiplicity result for problem ($\mathscr{P}$)

In this section, we will suppose that

$$e(t) = 0 , \quad \forall t \in \mathbb{R} .$$

We have the following multiplicity result.

THEOREM 3.1. *Suppose that assumptions* (j_0)–(j_6) *hold, and moreover* $j(t, \cdot)$ *is even for every* $t \in \mathbb{R}$, *i.e.,* $j(t, -x) = j(t, x)$ *for all* $(t, x) \in \mathbb{R} \times \mathbb{R}^N$. *Then* ($\mathscr{P}$) *has infinitely many pairs* $(u, -u)$ *of nonconstant T-periodic solutions.*

Proof. It is clear that the functional I defined in (2.3) (without e) is even. Since by (j_0), $j(t, 0) = 0$, then $I(0) = 0$. In the proof of Theorem 1.1, we have shown that I satisfies the Palais-Smale condition and the estimate stated in (2.11). In order to apply to I the symmetric version of the Mountain Pass Theorem, it suffices to prove that I satisfies the following condition: for all $k \in \mathbb{N}^*$, there exists a subspace E of H_T^1 with $dim(E) = k$ such that

$$I(u) \to -\infty \text{ as } \|u\| \to +\infty , \quad u \in E .$$

Let $k \in \mathbb{N}^*$ and let E be a subspace such that $dim(E) = k$.

By the equivalence of the norms on a finite dimensional subspace, we have

$$\|u\|^2 \leqslant C\|u\|_\infty^2 \quad \forall u \in E , \tag{3.20}$$

where $C = C(E) > 0$ is a constant.

Let

$$m := \inf_{\substack{\|u\|_\infty = 2\bar{r} \\ u \in E}} \int_{|u(t)| > \bar{r}} \alpha(t)|u(t)|^\nu \, dt . \tag{3.21}$$

It is clear that $m > 0$. By (3.20) and (j_6), we have

$$I(u) \leqslant \frac{C}{2} \|u\|_\infty^2 - \int_{|u(t)| > \bar{r}} \alpha(t)|u(t)|^\nu \, dt + \int_{|u(t)| \leqslant \bar{r}} j(t, u(t))\, dt .$$

Since the function $j(t, \cdot)$ satisfies the Lipschitz condition on bounded sets stated in (j_1), there exists a constant $c > 0$ such that

$$\int_{|u(t)| \leqslant \bar{r}} j(t, u(t))\, dt \leqslant c .$$

Consequently, the following estimate is valid

$$I(u) \leqslant \frac{C}{2}\|u\|_\infty^2 - \frac{\|u\|_\infty^\nu}{2^\nu \bar{r}^\nu}\int_{|u(t)|>\bar{r}} \alpha(t) 2^\nu \bar{r}^\nu \frac{|u(t)|^\nu}{\|u\|_\infty^\nu}\, dt + c$$

$$\leqslant \frac{C}{2}\|u\|_\infty^2 - \frac{m}{2^\nu \bar{r}^\nu}\|u\|_\infty^\nu + c\,, \quad \forall u \in E \setminus \{0\}\,.$$

Since $\nu > 2$, we deduce that $I(u) \to -\infty$ as $\|u\| \to +\infty$, $u \in E$.

Hence we are in a position to apply the nonsmooth version of Symmetric Mountain Pass Theorem which yields the desired conclusion. This completes the proof. □

References

1. Adly, S., Goeleven, D. and Motreanu, D. (1997), Periodic and homoclinic solutions for a class of unilateral problems, *Discrete and Continuous Dynamical Systems* 3: 579–590.
2. Chang, K.C. (1981), Variational methods for non-differentiable functionals and their applications to partial differential equations, *J. Math. Anal. Appl.* 80: 102–129.
3. Clarke, F.H. (1983), *Optimization and Nonsmooth Analysis*, John Wiley & Sons, New York.
4. Motreanu, D. (1995), Existence of critical points in a general setting, *Set-Valued Anal.* 3: 295–305.
5. Motreanu, D. and Naniewicz, Z. (1996), Discontinuous semilinear problems in vector-valued functions spaces, *Differ. Int. Equations* 9: 581–598.
6. Panagiotopoulos, P.D. (1983), Nonconvex energy function, hemivariational inequalities and substationarity principles, *Acta Mech.* 48: 160–183.
7. Panagiotopoulos, P.D. (1993), *Hemivariational Inequalities, Applications in Mechanics and Engineering*, Springer-Verlag, Berlin, Heidelberg.